MID-SEMESTER EXAMINATION, 2ND SEMESTER, M.MATH I YEAR, COMPLEX ANALYSIS, 2009-10

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Q1. Show that for $z, w \in \mathbb{C}$, we have
$|(w^n - z^n) - nz^{n-1}(w-z)| \leq n(n-1)|z-w|^2.(max(|z|,|w|)^{n-2})$
for $n \geq 2$.
Hence deduce that any power series can be differentiated term by term within its disc of convergence.

A1. If $w = z$, then the inequality holds trivially.
Assume that $w \neq z$. Then
$|(w^n - z^n) - nz^{n-1}(w-z)|$
$= |(w-z)(w^{n-1} + zw^{n-2} + \cdots + z^{n-1}) - nz^{n-1}(w-z)|$
$= |w-z|.|w^{n-1} + w^{n-2}z + \cdots + z^{n-1} - nz^{n-1}|$
$= |w-z|.|(w^{n-1} - z^{n-1} + z(w^{n-2} - z^{n-2}) + \cdots + z^{n-2}(w-z))|$
$= |w-z|^2.|(w^{n-2} + w^{n-3}z + \cdots + z^{n-2}) + z(w^{n-3} + w^{n-4}z + \cdots + z^{n-3}) + \cdots + z^{n-2})|$
$\leq |w-z|^2.(max(|z|,|w|)^{n-2}((n-1) + (n-2) + \cdots + 1)$
$= \frac{n(n-1)}{2}|z-w|^2.(max(|z|,|w|)^{n-2})$
$\leq n(n-1)|z-w|^2.(max(|z|,|w|)^{n-2})$.

Suppose that $\sum a_n z^n$ has radius of convergence R. Then we know that $\sum n a_n z^n$ has also radius of convergence R. Let $f(z) = \sum a_n z^n$. Then for some $|h| < \delta$ and $|z|, |z+h| < R$ we have,
$|\frac{f(z+h)-f(z)}{h} - \sum n a_n z^{n-1}|$
$\leq \sum |a_n| |\frac{(z+h)^n - z^n}{h} - n a_n z^{n-1}|$
$= \sum \frac{|a_n|}{|h|} |(z+h)^n - z^n - nz^{n-1}h|$
$\leq \sum \frac{|a_n|}{|h|} n(n-1)|h|^2.(\max(|z|,|z+h|)^{n-2})$ (using the proved inequality)
$= \sum |a_n||h| n(n-1) R^{n-2}$
$\leq |h| \sum |a_n| |n^2 R^{n-2}$.

Now, $\limsup(|a_n|)^{\frac{1}{n}} = \frac{1}{R}$, so $\limsup(|na_n|)^{\frac{1}{n}} = \frac{1}{R}$ and also $\limsup(|n^4 a_n|)^{\frac{1}{n}} = \frac{1}{R}$, so for some N, we have $n \geq N$, $n^2|a_n| \leq \frac{1}{R^n n^2}$.
Therefore, $\sum_{n \geq N} |a_n| n^2 R^{n-2} \leq \sum_{n \geq N} \frac{1}{R^2 n^2}$. So, the series $\sum |a_n| n^2 R^{n-2}|$ converges and so $|\frac{f(z+h)-f(z)}{h} - \sum n a_n z^n| \to 0$ as $|h| \to 0$. Hence, $f'(z) = \sum n a_n z^{n-1}$.

Q2. Prove that a non-constant analytic function has no local maximum for its modulus. Deduce that all its local minima are zero.

A2. The first part is the statement of Maximum Modulus Principle. Consult any book on complex analysis for the proof.
Suppose, z_0 is a local minimum for its modulus, where the value is not 0. Then, we shall get a neighbourhood of z_0, where the function is non-zero. Then, if we consider the function $\frac{1}{f(z)}$, we shall get an analytic function on the open set, which has a local maxima at z_0, which is clearly a contradiction to the maximum Modulus Principle. Hence, all its local minima are zero.

Q3. Prove ab initio that there is a $\theta > 0$ such that $e^{i\theta} = 1$.

A3. $e^{i\theta} = 1$ implies $\cos\theta = 1$, $\sin\theta = 0$. Then we have $\theta = 2\pi$ for which $\cos\theta = 1$, $\sin\theta = 0$ or $e^{i\theta} = 1$.

Q4. If Ω is a convex domain then show that every holomorphic function on Ω has an anti-derivative.

A4. In fact, the result is well-known for a simply connected region. As a convex domain is a simply connected region, we have the result. For the proof for a simply connected region, see p.119, Complex Analysis by Serge Lang, Third Edition.

Q5. If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is analytic then show that f is a rational function.

A5. Let f be a non-constant function. Since, f is meromorphic, so $f(z)$ and $f(\frac{1}{z})$ have finitely many zeroes and poles (each of them is of finite order) on the closed unit disc. And as the zeroes and poles of $f(z)$ outside the open unit disc correspond bijectively to the poles and zeroes of $f(\frac{1}{z})$ on the closed unit disc, so $f(z)$ have finitely many zeroes and poles, each is of finite order.
Let $a_1, a_2, \cdots, a_k$ be the zeroes of f with corresponding orders $n_1, n_2, \cdots, n_k$. Let $b_1, \cdots, b_l$ be the poles of f with corresponding orders $m_1, \cdots, m_l$.
Let $g = \frac{\prod_1^k (z-a_i)^{n_i}}{\prod_1^l (z-b_j)^{m_j}}$.
Let $h = \frac{f}{g}$. So, $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is function, which has no zeroes and poles. So, h extends to a non-zero bounded entire function, so $h = c$, where $c \in \mathbb{C} - \{0\}$. Therefore, $f = c.\frac{\prod_1^k (z-a_i)_i^n}{\prod_1^l (z-b_j)_j^m}$. Hence, f is a rational function.